Symmetry constraints for real dispersionless Veselov-Novikov equation

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Abstract

Symmetry constraints for dispersionless integrable equations are discussed. It is shown that under symmetry constraints the dispersionless Veselov-Novikov equation is reduced to the $1+1$-dimensional hydrodynamic type systems.

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1 Introduction.

Dispersionless integrable equations have attracted recently a considerable interest (see e.g. [1]-[5]). They arise in various fields of physics, mathematical physics and applied mathematics. Several methods and approaches have been used to study dispersionless systems, from the quasi-classical Lax pair representation with its close relationship with the Whitham universal hierarchy [2, 3] to the quasi-classical version of the inverse scattering method. In particular, the quasi-classical $\bar{\partial}$-dressing method [6, 7, 8], recently formulated, is a general and systematic approach to construct dispersionless

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integrable systems and to find their solutions. On the other hand a reduction method (see e.g. [9, 10, 11]) provides us also with the effective way to solve $2+1$-dimensional dispersionless equations. It was shown [12] that symmetry constraints for dispersionless equations provide us with an efficient way to construct reductions. In [12] certain hydrodynamic type reductions of dispersionless Kadomtsev-Petviashvili (dKP) and dispersionless two-dimensional Toda Lattice (2DdTL) equations have been constructed using symmetry constraints.

In this paper we study symmetry constraints for dispersionless Veselov-Novikov equation (dVN) and analyze the corresponding hydrodynamic type equations.

In section 2 we remind the definition of *symmetry constraint*. In sections 3 and 4 symmetry constraints for soliton (dispersive) and dispersionless equations respectively are considered, where the Kadomtsev-Petviashvili (KP) and dKP equation are considered as examples. Symmetry constraints for real dVN equation are discussed in section 5. In section 6 we demonstrate how symmetry constraints for dVN equation allow us to reduce this $2+1$-dimensional equation to $1+1$-dimensional hydrodynamic type systems.

2 Symmetry constraints.

Let us consider a partial differential equation for the scalar function $u = u(t) = u\left(t_1, t_2, \dots\right)$

$$F\left(u, u_{t_i}, u_{t_i t_j}, \dots\right) = 0, \tag{1}$$

where $u_{t_i} = \partial u / \partial t_i$. By definition, a symmetry of equation (1) is a transformation $u(t) \to u'(t')$, such that $u'(t')$ is again a solution of (1) (for more details see e.g. [13]). Infinitesimal continuous symmetry transformations

$$t_i' = t_i + \delta t_i; \qquad u' = u + \delta u = u + \epsilon u_\epsilon. \tag{2}$$

are defined by the linearized equation (1) [13]

$$L\delta u = 0, \tag{3}$$

where L is the Gateaux derivative of F

$$L\delta u := \frac{dF}{d\epsilon}\left(u + \epsilon u_\epsilon, \frac{\partial}{\partial t_i}\left(u + \epsilon u_\epsilon\right), \dots), \dots\right)\Bigg|_{\epsilon=0}. \tag{4}$$

Any linear superposition $\delta u = \sum_k c_k \delta_k u$ of infinitesimal symmetries $\delta_k u$ is, obviously, an infinitesimal symmetry. By definition a *symmetry constraint* is a requirement that certain superposition of infinitesimal symmetries vanishes, i.e.

$$\sum_k c_k \delta_k u = 0. \tag{5}$$

Since null function is a symmetry of equation (1), the constraint (5) is compatible with equation (1). Symmetry constraints allow us to select a class of solutions which possess some invariance properties. For instance, well-known symmetry constraint $\delta u = \epsilon u_{t_k} = 0$, selects solutions which are stationary with respect to the "time" t_k.

3 Soliton equations.

Symmetry constraints for $2+1$-dimensional soliton equations have been discussed the first time in the papers [14, 15]. Here, we discuss the KP equation equation (see e.g. [16])

$$\begin{aligned} u_t &= \frac{3}{2} u u_x + u_{xxx} + \frac{3}{4}\omega_y \\ \omega_x &= u_y, \end{aligned} \tag{6}$$

where $x := t_1$, $y := t_2$ and $t := t_3$. KP equation (6) is equivalent to the compatibility of the following linear problems [16]

$$\begin{aligned} \psi_y &= \psi_{xx} + u\psi \\ \psi_t &= \psi_{xxx} + \frac{3}{2} u \psi_x + \left(\frac{3}{2} u_x + \frac{3}{4}\omega \right) \psi. \end{aligned} \tag{7}$$

The symmetries equation (3) for KP assumes the form

$$(\delta u)_t = \frac{3}{2} \left(u_x \delta u + u(\delta u)_x \right) + (\delta u)_{xxx} + \frac{3}{4} (\delta \omega)_y, \tag{8}$$

$$(\delta \omega)_x = (\delta u)_y. \tag{9}$$

Now, introducing the adjoint linear problems of (7) defined as

$$\begin{aligned} -\psi^*_y &= \psi^*_{xx} + u\psi^* \\ \psi^*_t &= \psi^*_{xxx} + \frac{3}{2} u \psi^*_x + \left(\frac{3}{2} u_x - \frac{3}{4}\omega \right) \psi^*, \end{aligned} \tag{10}$$

one can verify directly that the function $\phi = (\psi\psi^*)_x$ obeys the linearized KP equation (8) [17], i.e. $(\psi\psi^*)_x$ is an infinitesimal symmetry of the KP equation. A class of symmetry constraint can be taken as

$$u_{t_n} = (\psi\psi^*)_x\,, \qquad n = 1, 2, 3. \tag{11}$$

The simplest of them is

$$u_x = (\psi\psi^*)_x\,, \tag{12}$$

which can be integrated to

$$u = \psi\psi^*. \tag{13}$$

Substituting (13) in the first equation of (7) and its adjoint (10), one obtains

$$\psi_y + \psi_{xx} + \psi^2\psi^* = 0 \tag{14}$$

$$-\psi^*_y + \psi^*_{xx} + (\psi^*)^2\psi = 0, \tag{15}$$

that is the AKNS system [16], which is reduced to the nonlinear Schrödinger equation if $\psi^* = \bar{\psi}$, where the “bar” means complex conjugation.

Substituting (13) in the second equations of linear problems and its adjoint and observing that $\omega = \psi^*_x\psi - \psi_x\psi^*$, one gets the higher AKNS system

$$\psi_t = 3\psi\psi^*\psi_x + \psi_{xxx} \tag{16}$$

$$\psi^*_t = 3\psi\psi^*\psi^*_x + \psi^*_{xxx}. \tag{17}$$

It is a straightforward check that if ψ and ψ^* obey equations (14)-(17), then $u = \psi\psi^*$ solves KP equation. Thus, symmetry constraints can be used to find solutions of $2+1$-dimensional system using solutions of the $1+1$-dimensional integrable systems.

4 Nonlinear dispersionless equations.

The dispersionless limit of soliton equations can be performed introducing slow variables, formally substituting $t_n \to \epsilon^{-1}t_n$, and looking for solutions having a certain behavior when $\epsilon \to 0$, for instance

$$u\left(\frac{t_n}{\epsilon}\right) \to u(t_n) + O(\epsilon), \qquad \epsilon \to 0. \tag{18}$$

For example, the dispersionless limit of KP equation is

$$
\begin{aligned}
u_t &= \frac{3}{2}uu_x + \frac{3}{4}\omega_y \\
\omega_x &= u_y.
\end{aligned}
\tag{19}
$$

The dispersionless limit of an integrable equation corresponds to the quasi-classical limit of the corresponding linear problems. In fact, representing the solution ψ of (7) as

$$
\psi = \psi_0 \exp\left(\frac{S}{\epsilon}\right), \tag{20}
$$

where $S\left(\lambda; \frac{x}{\epsilon}, \frac{y}{\epsilon}, \frac{t}{\epsilon}\right) \rightarrow S(\lambda; x, y, t) + O(\epsilon)$, and λ is the so-called *spectral parameter*, in the limit $\epsilon \rightarrow 0$ one gets from (7) the following pair of Hamilton-Jacobi type equations

$$
\begin{aligned}
S_y &= S_x^2 + u \\
S_t &= S_x^3 + \frac{3}{2}uS_x + \frac{3}{2}u_x + \frac{3}{4}\omega,
\end{aligned}
\tag{21}
$$

where $\omega_x = u_y$. The compatibility condition for (21) is just the dKP equation (19). Similarly to the dispersionfull case, we have the linearized dKP

$$
\begin{aligned}
(\delta u)_t &= \frac{3}{2}\left(u_x \delta u + u(\delta u)_x\right) + \frac{3}{4}(\delta\omega)_y, \\
(\delta\omega)_x &= (\delta u)_y,
\end{aligned}
\tag{22}
$$

whose solutions are infinitesimal symmetries of dKP.

Theorem 1 *Given any solutions S_i and $\tilde{S}_i$ of the Hamilton-Jacobi equations (21), the quantity $\delta u = \sum_{i=1}^{N} c_i \left(S_i - \tilde{S}_i\right)_{xx}$, where c_i are arbitrary constants, is a symmetry of dKP equation.*
Proof. It is straightforward to check that $\left(S_i - \tilde{S}_i\right)_{xx}$ satisfies equation (22). □

This type of symmetries has been introduced for the first time in [12], within the quasiclassical $\bar{\partial}$-dressing approach. A simple example of symmetry constraint for dKP, parallel to (12), is

$$
u_x = S_{xx}. \tag{23}
$$

Under this constraint the Hamilton-Jacobi system (21) gives rise [12] to the following hydrodynamic type system (the dispersionless nonlinear Schrödinger equation) [1]

$$\begin{aligned} \tilde{u}_y &= \left(\tilde{u}^2 + u\right)_x, \\ u_y &= 2\left(\tilde{u}u\right)_x, \end{aligned} \tag{24}$$

where $\tilde{u} = \partial S_x / \partial \lambda$.

5 Real dVN equation.

The Veselov-Novikov (VN) equation has been introduced as the two dimensional integrable extension of KdV in 1984 [18]. It looks like

$$u_t = (uV)_z + \left(u\bar{V}\right)_{\bar{z}} + u_{zzz} + u_{\bar{z}\bar{z}\bar{z}} \tag{25}$$

$$V_{\bar{z}} = -3u_z, \tag{26}$$

where $z = x + iy$. It is equivalent to the compatibility condition for equations

$$\psi_{z\bar{z}} = u\psi \tag{27}$$

$$\psi_t = \psi_{zzz} + \psi_{\bar{z}\bar{z}\bar{z}} + (V\psi_z) + \left(\bar{V}\psi_{\bar{z}}\right). \tag{28}$$

The VN equation has applications in differential geometry [19, 20]. Recently, it was shown that the dVN equation governs the propagation of light in a special class of nonlinear media in the limit of geometrical optics [21].

The dVN equation can be obtained as slow variables expansion of the VN equation (25). Setting $\psi{=}\psi_0(\lambda,\epsilon^{-1}z,\epsilon^{-1}\bar{z},\epsilon^{-1}t)\exp\epsilon^{-1}S(\lambda,z,\bar{z},t)$ just like in the previous section, one has the following pair of Hamilton-Jacobi equations [2, 8]

$$S_z S_{\bar{z}} = u, \tag{29}$$

$$S_t = S_z^3 + S_{\bar{z}}^3 + VS_z + \bar{V}S_{\bar{z}}, \tag{30}$$

and the equation

$$\begin{aligned} u_t &= (uV)_z + \left(u\bar{V}\right)_{\bar{z}} \\ V_{\bar{z}} &= -3u_z. \end{aligned} \tag{31}$$

In his paper we consider the case of real-valued u.
Linearized version of (31) is of the form

$$\begin{aligned}(\delta u)_t &= (V\delta u + u\delta V)_z + \left(\bar{V}\delta u + \delta\bar{V}u\right)_{\bar{z}} \\ V_{\bar{z}} &= -3u_z; \quad (\delta V)_{\bar{z}} = -3\,(\delta u)_z\,. \end{aligned} \tag{32}$$

Theorem 2 *Given any solutions S_i and $\tilde{S}_i$ of the Hamilton-Jacobi equations (29)-(30), the quantity*

$$\delta u = \sum_{i=1}^{N} c_i \left(S_i - \tilde{S}_i\right)_{z\bar{z}}, \tag{33}$$

where c_i are arbitrary constants, is a symmetry of dVN equation.
Proof. It is straightforward to check that $\left(S_i - \tilde{S}_i\right)_{z\bar{z}}$ satisfies equation (32).
□

In particular, one can choose $S_i = S(\lambda = \lambda_i)$ and $\tilde{S}_i = S(\lambda = \lambda_i + \mu_i)$. In the case $\mu_i \to 0$ and $c_i = \tilde{c}_i/\mu_i$, one has the class of symmetries given by

$$\delta u = \sum_{i=1}^{N} \tilde{c}_i \phi_{iz\bar{z}} \tag{34}$$

$$\phi_i = \frac{\partial S}{\partial \lambda}(\lambda = \lambda_i). \tag{35}$$

In what follows we will discuss three particular cases of real reductions, providing real solutions of dVN.

If S is a solution of Hamilton-Jacobi equations (29), then $-\bar{S}$ is a solution as well. Thus, for real-valued S ($S = \bar{S}$), specializing constraint (33) for $N = 1$, we have a simple constraint

$$\text{Case I} \qquad\qquad u_x = (S)_{z\bar{z}}\,. \tag{36}$$

For complex valued S one has the constraint

$$\text{Case II} \qquad\qquad u_x = \frac{1}{2}\left(S + \bar{S}\right)_{z\bar{z}}\,. \tag{37}$$

The last example of constraint is nothing but a particular case of (34), *i.e.*

$$\text{Case III} \qquad\qquad u_x = \phi_{z\bar{z}}. \tag{38}$$

6 Hydrodynamic type reductions of the dVN equation.

6.1 Case I

Let us introduce the functions $\rho_1 := S_x$ and $\rho_2 := S_y$. Thus, the symmetry constraint (36) can be written as follows

$$u_x = \frac{1}{4}\left(S_{xx} + S_{yy}\right) = \frac{1}{4}\left(\rho_{1x} + \rho_{2y}\right). \tag{39}$$

In order to analyze constraint (39) it is more convenient to consider equations (29) in Cartesian coordinates (x, y), i.e.

$$S_x^2 + S_y^2 = 4u \tag{40}$$

$$S_t = \frac{1}{4}S_x^3 - \frac{3}{4}S_x S_y^2 + V_1 S_x + V_2 S_y, \tag{41}$$

where $V = V_1 + iV_2$, while dVN equation acquires the form

$$u_t = (uV_1)_x + (uV_2)_y \tag{42}$$

$$V_{1x} - V_{2y} = -3u_x \tag{43}$$

$$V_{2x} + V_{1y} = 3u_y. \tag{44}$$

Substituting (40) into (39), one gets the following hydrodynamic type system

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_y = \begin{pmatrix} 0 & 1 \\ 2\rho_1 - 1 & 2\rho_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_x. \tag{45}$$

Now, let us focus on definition $V_{\bar{z}} := -3u_z$. Differentiating it with respect to x, using constraint (36) and equations (45), one obtains the equations

$$V_{1x} = -\frac{3}{2}\rho_{1x} + \frac{3}{4}\left(\rho_1^2 + \rho_2^2\right)_x \tag{46}$$

$$V_{2x} = \frac{3}{2}\rho_{2x}, \tag{47}$$

which can be trivially integrated providing the following explicit formulas for V_1 and V_2 in terms of ρ_1 and ρ_2:

$$V_1 = -\frac{3}{2}\rho_1 + \frac{3}{4}\left(\rho_1^2 + \rho_2^2\right)$$

$$V_2 = \frac{3}{2}\rho_2. \tag{48}$$

At this point we can derive t-dependent equations for ρ_1 and ρ_2. Differentiating equation (41) and using (45) and (48), one obtains the system

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_t = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}_x, \tag{49}$$

where

$$\begin{aligned} A_{11} &= 3\rho_1 (\rho_1 - 1), & A_{12} &= 3\rho_2, \\ A_{21} &= 3\rho_2 (2\rho_1 - 1), & A_{22} &= 3\rho_1 (\rho_1 - 1) + 6\rho_2^2. \end{aligned}$$

6.2 Case II

In this case (presenting the complex-valued function S in terms of its real and imaginary parts, $S = \rho + i\varphi$) the symmetry constraint (37) acquires the form

$$u_x = \frac{1}{4} (\rho_{xx} + \rho_{yy}). \tag{50}$$

Equation (40) is equivalent to the system

$$(\nabla\rho)^2 - (\nabla\varphi)^2 = 4u \tag{51}$$

$$\nabla\rho \cdot \nabla\varphi = 0, \tag{52}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and notation $\nabla\rho := (\rho_1, \rho_2)$ and $\nabla\varphi := (\varphi_1, \varphi_2)$ is introduced. Let us note that equation (52) allows to express, for instance, the component φ_2 in terms of the other ones ($\varphi_2 = -\rho_1\varphi_1/\rho_2$), so that only the functions ρ_2, ρ_2 and φ_1 are independent. By using the constraint (50), similar to the previous case, one shows that ρ_1, ρ_2, and φ_1, satisfy the hydrodynamic system

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_y = \begin{pmatrix} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_x, \tag{53}$$

where

$$\begin{aligned} a_1 &= 2\rho_1 \left(1 - \frac{\varphi_1^2}{\rho_2^2}\right) - 1, \quad a_2 = 2\left(\rho_2 + \frac{\rho_1^2\varphi_1^2}{\rho_2^3}\right), \\ a_3 &= -2\varphi_1 \left(1 + \frac{\rho_1^2}{\rho_2^2}\right), \quad b_1 = -\frac{\varphi_1}{\rho_2}, \quad b_2 = \frac{\rho_1}{\rho_2^2}\varphi_1, \quad b_3 = -\frac{\rho_1}{\rho_2}. \end{aligned}$$

Just like in the previous section, starting with the definition of V and differentiating it with respect to x, it possible to express its real and imaginary parts in terms of ρ_1, ρ_2, φ_1 and φ_2

$$\begin{aligned} V_1 &= -\frac{3}{2}\rho_1 + \frac{3}{4}\left(\rho_1^2 + \rho_2^2 - \varphi_1^2 - \varphi_2^2\right) \\ V_2 &= \frac{3}{2}\rho_2. \end{aligned} \tag{54}$$

or

$$\begin{aligned} V_1 &= -\frac{3}{2}\rho_1 + \frac{3}{4}\left(\rho_1^2 + \rho_2^2 - \varphi_1^2 - \frac{\rho_1^2\varphi_1^2}{\rho_2^2}\right) \\ V_2 &= \frac{3}{2}\rho_2. \end{aligned}$$

Separating real and imaginary parts in equation (41), one gets the system

$$\rho_t = \frac{1}{4}\left(\rho_x^3 - 3\rho_x\varphi_x^2\right) - \frac{3}{4}\left(\rho_x\rho_y^2 - \rho_x\varphi_y^2 - 2\rho_y\varphi_x\varphi_y\right) + V_1\rho_x + V_2\rho_y, \tag{55}$$

$$\varphi_t = \frac{1}{4}\left(-\varphi_x^3 + 3\rho_x^2\varphi_x\right) - \frac{3}{4}\left(2\rho_x\rho_y\varphi_y + \varphi_x\rho_y^2 - \varphi_x\varphi_y^2\right) + V_1\varphi_x + V_2\varphi_y. \tag{56}$$

Substituting expressions (54) into (55) and (56) and differentiating with respect to x and y, one obtains the hydrodynamic type system for ρ_1, ρ_2 and φ_1

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_t = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \varphi_1 \end{pmatrix}_x, \tag{57}$$

where

$$\begin{aligned} &B_{11} = 3\left(\rho_1^2 - \varphi_1^2\right) - \frac{3}{2}\rho_1, \quad B_{12} = 0, \quad B_{13} = -3\rho_1\varphi_1, \\ &B_{21} = \rho_2\left(6\rho_1 - 3\right) - \frac{9\rho_1}{\rho_2}\varphi_1^2, \quad B_{22} = 3\left(\rho_1\left(\rho_1 - 1\right) + 2\rho_2^2 - \varphi_1^2\right), \\ &B_{23} = -6\rho_2\varphi_1, \quad B_{31} = \frac{3}{2}\varphi_1\left(4\rho_1 - 1\right), \quad B_{32} = \frac{3}{2}\frac{\rho_1^2}{\rho_2}\varphi_1\left(\rho_2 + 1\right), \\ &B_{33} = 3\left(\rho_1^2 - \varphi_1^2\right) - \frac{3}{2}\rho_1. \end{aligned}$$

6.3 Case III

Let us note that symmetry constraint (38) implies that function ϕ must be real-valued, and we denote $(\sigma_1, \sigma_2) := \nabla\phi$. Hence, the symmetry constraint (38) looks like

$$u_x = \frac{1}{4}\left(\sigma_{1x} + \sigma_{2y}\right). \tag{58}$$

Moreover, for sake of simplicity, we assume function S to be real-valued as well, and denote $(\rho_1, \rho_2) := \nabla S$. Differentiating equation (40) with respect to λ, we obtain the algebraic relation

$$\rho_1\sigma_1 + \rho_2\sigma_2 = 0, \tag{59}$$

which allows us to eliminate, for instance, ρ_2. Using these assumptions, we obtain the following hydrodynamic type system in the variables x and y, for the functions σ_1, σ_2 and ρ_1

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \end{pmatrix}_y = \begin{pmatrix} 0 & 1 & 0 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \end{pmatrix}_x \tag{60}$$

where

$$\begin{aligned} c_1 &= 2\frac{\sigma_1\rho_1^2}{\sigma_2^2} - 1, \quad c_2 = -2\frac{\sigma_1^2\rho_1^2}{\sigma_2^3}, \quad c_3 = 2\rho_1\left(1 + \frac{\sigma_1^2}{\sigma_2^2}\right), \\ d_1 &= -\frac{\rho_1}{\sigma_2}, \quad d_2 = \frac{\sigma_1}{\sigma_2^2}\rho_1, \quad d_3 = -\frac{\sigma_1}{\sigma_2}. \end{aligned} \tag{61}$$

Using equation (41), the corresponding equation for ϕ, obtained by differentiation of (41) with respect to λ and the system (60), one gets the following expressions of V_1 and V_2

$$\begin{aligned} V_1 &= -\frac{3}{2}\sigma_1 + \frac{3}{4}\left(\rho_1^2 + \rho_2^2\right), \\ V_2 &= \frac{3}{2}\sigma_2. \end{aligned} \tag{62}$$

Expressing ρ_2 in terms of σ_1, σ_2 and ρ_1, one gets

$$V_1 = -\frac{3}{2}\sigma_1 + \frac{3}{4}\rho_1^2 + \frac{3}{4}\frac{\rho_1^2\sigma_1^2}{\sigma_2^2} \tag{63}$$

$$V_2 = \frac{3}{2}\sigma_2. \tag{64}$$

Using the formula (63), one obtains

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \end{pmatrix}_t = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \rho_1 \end{pmatrix}_x \tag{65}$$

where

$$\begin{aligned} & C_{11} = 3\left(\frac{3}{2}\rho_1^2 - \sigma_1\right), \quad C_{12} = 3\sigma_2, \quad C_{13} = 9\rho_1\sigma_1, \\ & C_{21} = \frac{3\sigma_1^2}{\sigma_2^2}\rho_1^4\left(\frac{1}{\sigma_2} - 1\right) + \frac{3}{2}\sigma_1\rho_1^2\left(1 - \frac{3}{\sigma_2}\right) - 3\sigma_2, \\ & C_{22} = \frac{3}{2}\rho_1^2\left(3 + 2\frac{\sigma_1^2}{\sigma_2^2}\right) - 3\sigma_1, \quad C_{23} = 3\rho_1\sigma_2\left(2 + \frac{\sigma_1^2}{\sigma_2^2}\right) \\ & C_{31} = -3\rho_1, \quad C_{32} = 0, \quad C_{33} = 3\left(\rho_1^2 - \sigma_1\right). \end{aligned}$$

Physical and geometrical meanings of the hydrodynamic type systems obtained in this paper will be discussed elsewhere.

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